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The Spectrum of a Periodic Complex Jacobi Matrix Revisited

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Let G be a complex periodic Jacobi matrix of period k. We reduce the study of the spectrum of G to that of a block tridiagonal matrix of the form

$$\begin{pmatrix} O_k & I_k & O_k & \cdots \\ I_k & O_k & I_k & \cdots \\ O_k & I_k & O_k & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where I_k and O_k denote the identity and null matrices of order k, respectively.

1. INTRODUCTION

In the past few years there has been a growing interest in the study of the spectral properties of non-symmetric operators defined by band matrices with complex coefficients. In this regard see [1–6]. In particular, this is due to numerous applications in the theory of continued fractions, Padé and Hermite–Padé approximation. Complex periodic Jacobi matrices are particularly important. The complement of the essential spectrum of the operator associated to any such matrix determines, except for isolated points, the region of convergence of the Chebyshev continued fractions whose parameters are asymptotically periodic and the limits coincide with the elements of the periodic Jacobi matrix. In this connection, using analytic methods, the spectrum of a complex periodic Jacobi matrix was investigated in [6]. The study of real periodic Jacobi matrices was initiated by J. L. Geronimus in [9] and subsequently developed in [8, 11]. In [4], complex asymptotically periodic Jacobi matrices with real limits were considered. We present a new approach to the study of the spectrum of a complex periodic Jacobi matrix based primarily on an algebraic relation which

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allows to reduce this study to that of a very simple block tridiagonal matrix (see Theorem 1 and (12) below).

Let

$$G=\begin{pmatrix} b^{(0)} & a^{(1)} & 0 & \cdots \\ a^{(1)} & b^{(1)} & a^{(2)} & \cdots \\ 0 & a^{(2)} & b^{(2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1}$$

where $\{a^{(n)}\}_{n\geqslant 1}$, $\{b^{(n)}\}_{n\geqslant 1}$ are k-periodic sequences of complex numbers. That is

$$a^{(n)}=a^{(j)}, \qquad b^{(n)}=b^{(j)}, \qquad j=1, ..., k, \tag{2}$$

for all $n=mk+j$ and $m=0, 1, ...,$ $b^{(0)}=b^{(k)}$. This matrix defines a bounded linear operator on ℓ^2 by the usual operation of multiplication of a matrix times a vector. We also denote this operator by G.

For each fixed $m=0, 1, ...,$ consider the sequence $\{R_n^{(m)}\}$, $n=0, 1, ...,$ of monic polynomials generated by the three term recurrence relation

$$R_{n+1}^{(m)}(z)=(z-b^{(n+m)})\, R_n^{(m)}(z)-[a^{(n+m)}]^2\, R_{n-1}^{(m)}(z), \tag{3}$$

with initial conditions

$$R_{-1}^{(m)}(z)=0, \qquad R_0^{(m)}(z)=1.$$

Set

$$Q_k(z)=(a^{(1)}\cdots a^{(k)})^{-1}\,[R_k^{(0)}(z)-[a^{(k)}]^2\, R_{k-2}^{(1)}(z)]. \tag{4}$$

Because of the banded structure of G, any linear combination of powers of G (that is, a polynomial evaluation of G), also defines a bounded linear operator on ℓ^2. We have

THEOREM 1. *Let G be the matrix defined by* (1)–(2) *and let Q_k be the polynomial given by* (4). *Then*

$$Q_k(G)=\begin{pmatrix} A_k & I_k & O_k & \cdots \\ I_k & O_k & I_k & \cdots \\ O_k & I_k & O_k & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5}$$

where A_k, I_k, and O_k denote matrices of order k, with I_k the identity and O_k the null matrix.

From Theorem 1, using standard results of operator theory, one can easily characterize the spectrum of G and find its essential spectrum. In the sequel, $\sigma(\cdot)$, $\sigma_{ess}(\cdot)$, and $\sigma_p^f(\cdot)$ denote respectively the spectrum, the essential spectrum, and the set of eigenvalues of finite type of the operator $(\cdot)$ (for the definitions see, for example, [10]).

THEOREM 2. *The spectrum of G satisfies*

$$Q_k(\sigma(G)) = \sigma(Q_k(G)) = \sigma_p^f(Q_k(G)) \cup [-2, 2]. \tag{6}$$

In particular, for the essential spectrum of G, we have

$$\sigma_{ess}(G) = \{z : Q_k(z) \in [-2, 2]\}, \tag{7}$$

and $\sigma_p^f(Q_k(G))$ consists of isolated points in $\mathbb{C} \backslash [-2, 2]$.

As stated above, Theorem 2 was proved in [6] following a different approach. The structure of the paper is as follows. In Section 2, we deduce Theorem 2 from Theorem 1. Section 3 is devoted to the proof of Theorem 1. In the sequel, we preserve the notations introduced above.

2. THE SPECTRUM OF THE OPERATOR G

We start out pointing out some immediate consequences from known results from operator theory. Let p be a non constant polynomial. From the Spectral Mapping Theorem (see [12, p. 53; 7]) it follows that

$$p(\sigma(G)) = \sigma(p(G)), \tag{8}$$

$$p(\sigma_{ess}(G)) = \sigma_{ess}(p(G)). \tag{9}$$

Moreover, Weyl's Theorem (see [10]) states that

$$\sigma_{ess}(H + K) = \sigma_{ess}(H), \tag{10}$$

if H is normal and K is an arbitrary compact operator.

Proof of Theorem 2. Using (9) with $p = Q_k$, (5), and (10) it follows that

$$Q_k(\sigma_{ess}(G)) = \sigma_{ess}(Q_k(G)) = \sigma_{ess}(H), \tag{11}$$

where

$$H=\begin{pmatrix} O_k & I_k & O_k & \cdots \\ I_k & O_k & I_k & \cdots \\ O_k & I_k & 0_k & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{12}$$

Consider two column vectors $v=(v_1, v_2, ...)^t$ and $w=(w_1, w_2, ...)^t$ (not necessarily in ℓ^2). In the sequel, $(\cdot)^t$ denotes the transpose of the vector or matrix $(\cdot)$. Suppose that formally, we have that

$$(H-\lambda I)\, v=w.$$

This means that the coordinates of v and w are related through the equations

$$v_{-k+j}-\lambda v_j+v_{k+j}=w_j, \qquad j=1, 2, ...,$$

where $v_{-k+1}=\cdots=v_0=0$. From this it is easy to see that the spectrum and the point spectrum of H and the Jacobi matrix T with constant coefficients $a^{(i)}=1$, $b^{(i)}=0$, $i=0, 1, ...$ coincide. It is well known that the spectrum of T is $[-2, 2]$ and that it has no eigenvalues. Thus, due to (11),

$$Q_k(\sigma_{ess}(G))=\sigma_{ess}(Q_k(G))=[-2, 2],$$

and (7) follows.

Relation (6) is a consequence of (8) and the fact that since $\mathbb{C}\backslash\sigma_{ess}(Q_k(G))$ is connected, the set $\sigma(Q_k(G))\backslash\sigma_{ess}(Q_k(G))$ consists of isolated points in $\mathbb{C}\backslash\sigma_{ess}(Q_k(G))$ which are eigenvalues of finite type (see [10, Corollary 8.5]). Thus (6) takes place with which we conclude the proof. ∎

3. PROOF OF THEOREM 1

Notice that for each $m=0, 1, ...,$ the m-th power G^m of G is a symmetric (in the sense that $(G^m)^t=G^m$) $2m+1$-diagonal matrix. Therefore, for $0\leqslant m\leqslant k$, G^m may be written as a block tridiagonal matrix of the form

$$G^m=\begin{pmatrix} s^{(m)} & (x^{(m)})^t & O_k & \cdots \\ x^{(m)} & y^{(m)} & (x^{(m)})^t & \cdots \\ O_k & x^{(m)} & y^{(m)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{13}$$

where $x^{(m)}$, $y^{(m)}$ and $s^{(m)}$ are matrices of order k, $y^{(m)}$ is symmetric and $x^{(m)}$ is upper triangular. Moreover, the elements on the main diagonal of $x^{(m)}$ are zero except possibly for $m=k$. In particular, for $m=1$, we have that

$$x^{(1)} = \begin{pmatrix} 0 & \cdots & 0 & a^{(k)} \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad y^{(1)} = \begin{pmatrix} b^{(k)} & a^{(1)} & 0 & \cdots \\ a^{(1)} & b^{(1)} & a^{(2)} & \cdots \\ 0 & a^{(2)} & b^{(2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{14}$$

Consider the matrix of order k

$$J = x^{(1)} + y^{(1)} + (x^{(1)})^t.$$

Let us prove that for each $0 \leqslant m \leqslant k$

$$J^m = x^{(m)} + y^{(m)} + (x^{(m)})^t. \tag{15}$$

For $m=1$, this formula is true by the definition of J. Assume that the formula holds for $0 \leqslant m \leqslant k-1$, let us prove that it is also satisfied for $m+1$. In fact, since $G^{m+1} = G^m \cdot G$, direct calculations show that

$$\begin{aligned} x^{(m+1)} &= x^{(m)}y^{(1)} + y^{(m)}x^{(1)}, \\ y^{(m+1)} &= x^{(m)}(x^{(1)})^t + y^{(m)}y^{(1)} + (x^{(m)})^t\, x^{(1)}, \\ (x^{(m+1)})^t &= y^{(m)}(x^{(1)})^t + (x^{(m)})^t\, y^{(1)}. \end{aligned}$$

Thus

$$\begin{aligned} & x^{(m+1)} + y^{(m+1)} + (x^{(m+1)})^t \\ & \quad = x^{(m)}y^{(1)} + y^{(m)}x^{(1)} + x^{(m)}(x^{(1)})^t + y^{(m)}y^{(1)} \\ & \qquad + (x^{(m)})^t\, x^{(1)} + (y^{(m)})^t\, x^{(1)} + (x^{(m)})^t\, y^{(1)}. \end{aligned} \tag{16}$$

On the other hand, using the induction assumption, we have

$$J^{m+1} = J^m \cdot J = (x^{(m)} + y^{(m)} + (x^{(m)})^t)(x^{(1)} + y^{(1)} + (x^{(1)})^t).$$

Carrying out the products indicated by the parenthesis one obtains the same expression as on the right hand of (16) since $x^{(m)}x^{(1)} = (x^{(m)})^t\,(x^{(1)})^t = O_k$ because $x^{(m)}$ is upper triangular with all the elements on the main diagonal equal to zero.

Let p be a polynomial of degree $\leqslant k$. From what has been proven above it follows that

$$p(G)=\begin{pmatrix} s & x^t & O_k & \cdots \\ x & y & x^t & \cdots \\ O_k & x & y & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{17}$$

where x, y, and s are matrices of order k and

$$x+y+x^t=p(J). \tag{18}$$

Let G' denote the infinite dimensional periodic tridiagonal matrix which is obtained substituting $a^{(k)}$ by $-a^{(k)}$ in G. It is easy to check that

$$p(G')=\begin{pmatrix} s & -x^t & O_k & \cdots \\ -x & y & -x^t & \cdots \\ O_k & -x & y & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where x, y, and s denote the same matrices as in (17). Thus if

$$J'=-x^{(1)}+y^{(1)}-(x^{(1)})^t,$$

then

$$-x+y-x^t=p(J'). \tag{19}$$

The characteristic polynomials of J and J' are equal to

$$\det(zI_k-J)=q_k(z)-2a^{(1)}\cdots a^{(k)}, \qquad \det(zI_k-J')=q_k(z)+2a^{(1)}\cdots a^{(k)},$$

where $q_k(z)=R_k^{(0)}(z)-[a^{(k)}]^2\,R_{k-2}^{(1)}(z)$. This easily follows developing the determinants by their last row. From the Hamilton–Cayley Theorem, we obtain

$$q_k(J)=2a^{(1)}\cdots a^{(k)}I_k, \qquad q_k(J')=-2a^{(1)}\cdots a^{(k)}I_k.$$

Therefore, if we take $p=q_k$ in (18) and (19), we obtain the equations

$$x+y+x^t=2a^{(1)}\cdots a^{(k)}I_k, \qquad -x+y-x^t=-2a^{(1)}\cdots a^{(k)}I_k. \tag{20}$$

Summing these two equations, it follows that

$$2y=O_k. \tag{21}$$

On the other hand, since x is upper triangular, from (20) and (21), we also conclude that

$$x = a^{(1)} \cdots a^{(k)} I_k.$$

With this we conclude the proof of Theorem 1.

Remark 1. Due to the periodic structure of G (and $q_k(G)$) it may be shown that

$$A_k = \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,k-1} & 0 \\ \vdots & & & & \vdots \\ \gamma_{k-1,1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where $\gamma_{i,j}$ denote some complex numbers.

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